ON THE MOTION ABOUT A FIXED POINT OF A FAST SPINNING HEAVY SOLID

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The motion of a heavy solid about a fixed point with high initial angular velocity has been investigated (omitting the case of Lagrange) for the case of Goriachev-Chaplygin in [1,2]. Cases with restrictions on the location of the center of gravity, on moments of inertia and on the initial conditions has been investigated in [3,4].

In our work we apply the method of small parameters to investigate the periodic solutions of the equations of motion of a heavy solid with one point fixed rapidly spinning about one of the principal axes of the ellipsoid of inertia. In particular, we show that, with the exception of special cases, when $z_0 \neq 0$ the body will perform a pseudo-regular precession about the vertical axis in the first approximation, and that at least four of the six initial conditions are arbitrary.

1. Let us consider a heavy solid with one point fixed, whose ellipsoid of inertia is arbitrary, and whose center of gravity is arbitrarily located and not necessarily coinciding with the fixed point. General equations of motion of this solid and their first integrals are

$$\frac{dp}{dt} + A_1 qr = \frac{Mg}{A}(y_0\gamma'' - z_0\gamma'), \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma'' \qquad \begin{pmatrix} ABC, & pqr, \\ \gamma\gamma'\gamma'', & x_0y_0z_0 \end{pmatrix} \qquad (1.1)$$

$$\left(A_1 = \frac{C-B}{A},\ B_1 = \frac{A-C}{B},\ C_1 = \frac{B-A}{C}\right)$$

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma + y_0\gamma' + z_0\gamma'') =$$
$$= Ap_0^2 + Bq_0^2 + Cr_0^2 - 2Mg(x_0\gamma_0 + y_0\gamma_0' + z_0\gamma_0'')$$

$$Ap\gamma + Bq\gamma' + Cr\gamma'' = Ap_0\gamma_0 + Bq_0\gamma_0' + Cr_0\gamma_0''$$
$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1 \tag{1.2}$$

Here p_0, q_0, r_0, γ_0, γ_0' and γ_0'' are the intial values of the corresponding variables, symbols like (abc) mean cyclic permutations and indicate equations which are omitted.

We shall assume that at the initial instant of time the principal axis z of the ellipsoid of inertia makes an angle $\theta_0 (\theta_0 \neq 1/2\, k\pi,\ k = 0, 1, 2)$ with the vertical, and that the body spins about this principal axis with a high angular velocity r_0. Without loss of generality, we select the moving coordinate system in which the positive branches of the z-axis and of the x-axis do not make an obtuse angle with the direction of gravitational acceleration at the initial instant of time.

In this coordinate system, then $r_0 \gtrless 0$, and because of the restrictions on θ_0, the initial values γ_0 and γ_0'' must satisfy the conditions

$$\gamma_0 \geqslant 0, \qquad 0 < \gamma_0'' < 1 \tag{1.3}$$

Let us introduce

$$a = \frac{A}{C}, \qquad b = \frac{B}{C}, \qquad c^2 = \frac{Mgl}{C}, \qquad \mu = \frac{c\sqrt{\gamma_0''}}{r_0} \tag{1.4}$$
$$x_0 = lx_0', \qquad y_0 = ly_0', \qquad z_0 = lz_0', \qquad l^2 = x_0^2 + y_0^2 + z_0^2$$

By assuming that r_0 is large, we assume that μ is small. Let us now introduce a new variable through the formulas p_1, q_1, r_1, γ_1, γ_1', γ_1'' and τ

$$p = c\sqrt{\gamma_0''}\, p_1, \qquad q = c\sqrt{\gamma_0''}\, q_1, \qquad r = r_0 r_1 \tag{1.5}$$
$$\gamma = \gamma_0''\gamma_1, \qquad \gamma' = \gamma_0''\gamma_1, \qquad \gamma'' = \gamma_0''\gamma_1'', \qquad t = \tau / r_0$$

Equations (1.1) and their integrals (1.2) will, with the new variable, assume the form

$$\dot p_1 + A_1 q_1 r_1 = \mu a^{-1} (y_0'\gamma''_1 - z_0'\gamma_1'), \qquad \gamma_1 = r_1\gamma_1' - \mu q_1\gamma_1''$$
$$\dot q_1 + B_1 p_1 r_1 = \mu b^{-1} (z_0'\gamma_1 - x_0'\gamma_1''), \qquad \dot\gamma_1' = \mu p_1\gamma_1'' - r_1\gamma_1 \tag{1.6}$$
$$\dot r_1 = \mu^2 (- C_1 p_1 q_1 + x_0'\gamma_1' - y_0'\gamma_1), \qquad \dot\gamma_1'' = \mu (q_1\gamma_1 - p_1\gamma_1') \qquad (\dot u = du / d\tau)$$

$$r_1^2 = 1 + \mu^2 S_1 \quad \{S_1 = a (p_{10}^2 - p_1^2) + b (q_{10}^2 - q_1^2) -$$
$$- 2 [x_0' (\gamma_{10} - \gamma_1) + y_0' (\gamma_{10}' - \gamma_1') + z_0' (1 - \gamma_1'')\} \tag{1.7}$$

$$r_1\gamma_1'' = 1 + \mu S_2 \qquad \{S_2 = a (p_{10}\gamma_{10} - p_1\gamma_1) + b (q_{10}\gamma_{10}' - q_1\gamma_1)\} \tag{1.8}$$
$$\gamma_1^2 + \gamma_1'^2 + \gamma_1''^2 = (\gamma_0'')^{-2} \tag{1.9}$$

Using the first integrals (1.7) and (1.8), we express the variables r_1 and γ_1'' in terms of the remaining variables p_1, q_1, γ_1', γ_1'', and in those of their initial values p_{10}, q_{10}, γ_{10}, γ_{10}', and in those of the small parameter μ

$$r_1 = 1 + {}^1/_2\mu^2 [S_1 + 2z_0' (1 - \gamma_1'')] + \ldots$$
$$\gamma_1'' = 1 + \mu S_2 - {}^1/_2\mu^2 [S_1 + 2z_0' (1 - \gamma_1'')] + \ldots \qquad (1.10)$$

and reduce the remaining four equations of the system (1.6) to two second order equations

$$\ddot{p}_1 + \omega^2 p_1 = \mu [z_0' (a^{-1} + A_1 b^{-1}) \gamma_1 + A_1 b^{-1} x_0'] + \mu^2 \{- \omega^2 p_1 [S_1 + 2z_0' (1 - \gamma_1'')] + A_1 b^{-1} x_0' S_2 + A_1 C_1 p_1 q_1^2 - A_1 x_0' q_1 \gamma_1' - y_0' a^{-1} p_1 \gamma_1' + y_0' (A_1 + a^{-1}) q_1 \gamma_1 - z_0' a^{-1} p_1\} + \mu^3 z_0' \{{}^1/_2 (a^{-1} - b^{-1} A_1) [S_1 + 2z_0' (1 - \gamma_1'')] \gamma_1 - (2\omega^2 + a^{-1}) p_1 S_2\} + \ldots \qquad (1.11)$$

$$\ddot{\gamma}_1 + \gamma_1 = \mu (1 + B_1) p_1 + \mu^2 \{- \gamma_1 [S_1 + 2z_0' (1 - \gamma_1'')] + (1 + B_1) p_1 S_2 + (1 + C_1) p_1 q_1 \gamma_1' + x_0' \gamma_1'^2 - y_0' \gamma_1 \gamma_1' - b^{-1} z_0' \gamma_1 + b^{-1} x_0' - q_1^2 \gamma_1\} + \mu^3 [- z_0' (2 + b^{-1}) \gamma_1 S_2 + 2b^{-1} x_0' S_2] + \ldots \qquad (1.12)$$

$$\omega^2 = - A_1 B_1 = (a - 1) (b - 1) / ab = (A - C) (B - C) / AB \qquad (1.13)$$

Solving the first and fourth equation of system (1.6) for q_1 and γ_1' we obtain

$$q_1 = (A_1 r_1)^{-1} [- \dot{p}_1 + \mu a^{-1} (y_0' \gamma_1'' - z_0' \gamma_1')], \qquad \gamma_1' = (r_1)^{-1} [\dot{\gamma}_1 + \mu q_1 \gamma_1''] \qquad (1.14)$$

in which r_1 and γ_1'' are replaced by (1.10). Substituting (1.14) in the right-hand terms of equations (1.11) and (1.12), we obtain a quasilinear autonomous system with two degrees of freedom, whose right-hand sides depend on p_1, $\dot{p}_1$, γ_1, $\dot{\gamma}_1$, p_{10}, $\dot{p}_{10}$, γ_{10} and $\dot{\gamma}_{10}$.

We want to find periodic solutions of this system under conditions $A > B > C$, or $A < B < C$ (ω^2 is positive). In the first case, $\omega < 1$ and the z-axis should coincide with the major axis of the ellipsoid of inertia. In the second case, $\omega \geqslant 1$ and the z-axis should coincide with the minor axis of the ellipsoid of inertia. The case $\omega = 1$ corresponds to the disc

$$A + B = C \qquad (a + b = 1), \qquad z_0 = 0 \qquad (1.15)$$

Let us introduce new variables p_2 and γ_2 through

$$p_1 = p_2 + \mu\varkappa + \mu\varkappa_1\gamma_2, \qquad \gamma_1 = \gamma_2 + \mu\nu p_2 \qquad (1.16)$$

$$\varkappa = \frac{x_0' A_1}{b\omega^2}, \qquad \varkappa_1 = -\frac{z_0'}{1-\omega^2}\left(\frac{1}{a}+\frac{A_1}{b}\right), \qquad \nu = \frac{1+B_1}{1-\omega^2} \tag{1.17}$$

Using formulas (1.16) and (1.10) and (1.14) we can obtain the following power series expansion in μ:

$$S_i = S_{i1} + 2^{2-i}\mu S_{i2} + \ldots \qquad (i = 1, 2)$$

$$r_1 = 1 + {}^1/_2\mu^2 S_{11} + \ldots, \gamma_1'' = 1 + \mu S_{21} + \mu^2 (S_{22} - {}^1/_2 S_{11}) + \ldots \tag{1.18}$$

$$q_1 = -A^{-1}\dot{p}_2 + \mu A_1^{-1}(y_0' a^{-1} - \varkappa_2 \dot{\gamma}_2) + \ldots, \qquad \gamma_1' = \dot{\gamma}_2 + \mu\nu_2\dot{p}_2 + \ldots$$

$$\varkappa_2 = \varkappa_1 + a^{-1}z_0, \qquad \nu_2 = \nu - A_1^{-1} \tag{1.19}$$

where

$$S_{11} = a(p_{20}^2 - p_2^2) + b(\dot{p}_{20}^2 - \dot{p}_2^2)/A_1^2 - 2[x_0'(\gamma_{20} - \gamma_2) + y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2)] \tag{1.20}$$

$$S_{12} = a[\varkappa(p_{20} - p_2) + \varkappa_1(p_{20}\gamma_{20} - p_2\gamma_2)] - bA_1^{-2}[y_0' a^{-1}(\dot{p}_{20} - \dot{p}_2) - \varkappa_2(\dot{\gamma}_{20}\dot{p}_{20} - \dot{\gamma}_2\dot{p}_2)] - x_0'\nu(p_{20} - p_2) - y_0'\nu_2(\dot{p}_{20} - \dot{p}_2) + z_0' S_{21}$$

$$S_{21} = a(p_{20}\gamma_{20} - p_2\gamma_2) - bA_1^{-1}(\dot{p}_{20}\dot{\gamma}_{20} - \dot{p}_2\dot{\gamma}_2)$$

$$S_{22} = a[\nu(p_{20}^2 - p_2^2) + \varkappa(\gamma_{20} - \gamma_2) + \varkappa_1(\gamma_{20}^2 - \gamma_2^2)] + bA_1^{-1}[-\nu_2(\dot{p}_{20}^2 - \dot{p}_2^2) + a^{-1}y_0'(\dot{\gamma}_{20} - \dot{\gamma}_2) - \varkappa_2(\dot{\gamma}_{20}^2 - \dot{\gamma}_2^2)]$$

Substituting (1.16) and (1.18) into equations (1.11) and (1.12), we obtain

$$\ddot{p}_2 + \omega^2 p_2 = \mu^2 F(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu), \quad F = F_2 + \mu F_3 + \ldots \tag{1.21}$$

$$\ddot{\gamma}_2 + \gamma_2 = \mu^2\Phi(p_2, \dot{p}_2, \gamma_2, \dot{\gamma}_2, \mu) \quad \Phi = \Phi_2 + \mu\Phi_3 + \ldots$$

$$F_2 = f_2 - \nu\varkappa_1(1 - \omega^2)p_2$$

$$F_3 = f_3 - \varkappa_1\varphi_2 - \nu\varkappa_1\varkappa(1 - \omega^2) - \nu\varkappa_1^2(1 - \omega^2)\gamma_2 \tag{1.22}$$

$$\Phi_2 = \varphi_2 + \nu\varkappa(1 - \omega^2) + \nu\varkappa_1(1 - \omega^2)\gamma_2, \quad \Phi_3 = \varphi_3 - \nu f_2 + \nu^2\varkappa_1(1 - \omega^2)p_2$$

where

$$\tag{1.23}$$

$$f_2 = -\omega^2 p_2 S_{11} + A_1 b^{-1} x_0' S_{21} + C_1 A_1^{-1} p_2 \dot{p}_2^2 + x_0'\dot{p}_2\dot{\gamma}_2 - y_0' a^{-1} p_2 \dot{\gamma}_2 - y_0' A_1^{-1}(A_1 + a^{-1})\gamma_2\dot{p}_2 - z_0' a^{-1} p_2$$

$$f_3 = -\omega^2(\varkappa S_{11} + \varkappa_1\gamma_2 S_{11} + 2p_2 S_{12}) + A_1 b^{-1} x_0' S_{22} + C_1 A_1^{-1}[\varkappa\dot{p}_2^2 + \varkappa_1\gamma_2\dot{p}_2^2 - 2p_2\dot{p}_2(y_0' a^{-1} - \varkappa_2\dot{\gamma}_2)] - x_0'[-\nu_2\dot{p}_2^2 + \dot{\gamma}_2(y_0' a^{-1} - \varkappa_2\dot{\gamma}_2)] - y_0' a^{-1}[\dot{\gamma}_2(\varkappa + \varkappa_1\gamma_2) + \nu_2 p_2\dot{p}_2] + y_0' A_1^{-1}(A_1 + a^{-1})[\gamma_2(y_0' a^{-1} - \varkappa_2\dot{\gamma}_2) - \nu p_2\dot{p}_2] - z_0' a^{-1}(\varkappa + \varkappa_1\gamma_2) + {}^1/_2 z_0'(a^{-1} + A_1 b^{-1})\gamma_2 S_{11} + z_0' a^{-1} p_2 S_{21}$$

$$\varphi_2 = -\gamma_2 S_{11} + (1 + B_1)\, p_2 S_{21} - (1 - C_1)\, A_1^{-1} p_2 \dot{p}_2 \dot{\gamma}_2 + x_0' \dot{\gamma}_2^2 - y_0' \gamma_2 \dot{\gamma}_2 - \\ - z_0' b^{-1} \gamma_2 + x_0' b^{-1} - A_1^{-2} \gamma_2 \dot{p}_2^2$$

$$\varphi_3 = -\nu p_2 S_{11} - 2\gamma_2 S_{21} + (1 + B_1)\, p_2 S_{22} + (1 + B_1)(\varkappa + \varkappa_1 \gamma_2)\, S_{21} + \\ + (1 - C_1)\, A_1^{-1} [-(\varkappa + \varkappa_1 \gamma_2)\, \dot{p}_2 \dot{\gamma}_2 - \nu_2 p_2 \dot{p}_2^2 + p_2 \dot{\gamma}_2 (y_0' a^{-1} - \varkappa_2 \dot{\gamma}_2)] + \\ + 2x_0' \nu_2 \dot{p}_2 \dot{\gamma}_2 - y_0' (\nu p_2 \dot{\gamma}_2 + \nu_2 \gamma_2 \dot{p}_2) - z_0' b^{-1} \nu p_2 - z_0' b^{-1} \gamma_2 S_{21} + \\ + 2x_0' b^{-1} S_{21} - A_1^{-2} [-2\gamma_2 \dot{p}_2 (y_0' a^{-1} - \varkappa_2 \dot{\gamma}_2) + \nu p_2 \dot{p}_2^2]$$

This system has a first integral which can be obtained from the integral (1.9)

$$\gamma_2^2 + \dot{\gamma}_2^2 + 2\mu (\nu p_2 \gamma_2 + \nu_2 \dot{p}_2 \dot{\gamma}_2 + S_{21}) + \mu^2 (\ldots) = (\gamma_0'')^{-2} - 1 \qquad (1.24)$$

We are going to look for periodic solutions $p_2(\tau, \mu)$, $\dot{p}_2(\tau, \mu)$, $\gamma_2(\tau, \mu)$ and $\dot{\gamma}_2(\tau, \mu)$ of system (1.21), but only those which satisfy conditions

$$p_2(0, 0) = 0, \qquad \dot{p}_2(0, 0) = 0, \qquad \dot{\gamma}_2(0, \mu) = 0 \qquad (1.25)$$

Since system (1.21) is autonomous, the above conditions do not affect the generality of the solution.

Since the frequencies of the generating system

$$\ddot{p}_2 + \omega^2 p_2 = 0, \qquad \ddot{\gamma}_2 + \gamma_2 = 0 \qquad (1.26)$$

are ω and 1, we can construct [5] periodic solutions of system (1.21) in three different ways:

1) When the two frequencies are distinct but commensurate ($\omega = m/n$, where m and n are relative primes);

2) When the two frequencies are equal ($\omega = 1$);

3) When the two frequencies are noncommensurate (ω is irrational).

2. Let us consider the first case, $\omega = m/n$. In this case, there exist periodic solutions of the generating system (1.26) with the period $T_0 = 2\pi n$

$$p_2^{(0)} = M_1 \cos \omega\tau + M_2 \sin \omega\tau, \qquad \gamma_2^{(0)} = M_3 \cos \tau \qquad (2.1)$$

We assume that the initial autonomous system (1.21) has periodic solutions with the period $T_0 + \alpha$, which reduce to the generating solution (2.1) when $\mu = 0$. We shall write the initial conditions through the relations

$$p_2(0,\mu) = M_1 + \beta_1, \qquad \dot p_2(0,\mu) = \omega(M_2 + \beta_2)$$
$$\gamma_2(0,\mu) = M_3 + \beta_3, \qquad \dot\gamma_2(0,\mu) = 0 \tag{2.2}$$

Let us also introduce the operator

$$U = u + \frac{\partial u}{\partial M_1}\beta_1 + \frac{\partial u}{\partial M_2}\beta_2 + \frac{\partial u}{\partial M_3}\beta_3 + \frac{1}{2}\frac{\partial^2 u}{\partial M_1^2}\beta_1^2 + \ldots \qquad \left(\begin{matrix} U = G_k,\ H_k \\ u = g_k,\ h_k \end{matrix}\right)$$

and express [5] the periodic solution which we want to find in the form

$$p_2(\tau,\mu) = (M_1+\beta_1)\cos\omega\tau + (M_2+\beta_2)\sin\omega\tau + \sum_{k=2}^{\infty} G_k(\tau)\mu^k$$
$$\gamma_2(\tau,\mu) = (M_3+\beta_3)\cos\tau + \sum_{k=2}^{\infty} H_k(\tau)\mu^k \tag{2.3}$$

$$g_k(\tau) = \frac{1}{\omega}\int_0^\tau F_k'(t_1)\sin\omega(\tau - t_1)\,dt_1, \qquad h_k(\tau) = \int_0^\tau \Phi_k'(t_1)\sin(\tau - t_1)\,dt_1$$
$$F_k'(\tau) = \frac{1}{(k-2)!}\left(\frac{d^{k-2}F}{d\mu^{k-2}}\right)_{\beta=\mu=0}, \qquad \Phi_k'(\tau) = \frac{1}{(k-2)!}\left(\frac{d^{k-2}\Phi}{d\mu^{k-2}}\right)_{\beta=\mu=0} \tag{2.4}$$

Here β_1, $\omega\beta_2$ and β_3 are deviations of the initial values of p_2, $\dot p_2$ and γ_2 in the periodic solutions of equations (1.21) from the initial values of the same quantities in the generating solution (2.1). These deviations are functions of μ, and they vanish when $\mu = 0$.

Since the right-hand sides in system (1.21) begin from a term of the order μ^2, we have the following relations:

$$F_k'(\tau) = F_k(p_2^{(0)}, \dot p_2^{(0)}, \gamma_2^{(0)}, \dot\gamma_2^{(0)}) \equiv F_k^{(0)}$$
$$\Phi_k'(\tau) = \Phi_k(p_2^{(0)}, \dot p_2^{(0)}, \gamma_2^{(0)}, \dot\gamma_2^{(0)}) \equiv \Phi_k^{(0)} \qquad (k = 2,\ 3) \tag{2.5}$$

Let us find now the functions $F_2^{(0)}$ and $\Phi_2^{(0)}$. Introducing the notation

$$E = \sqrt{M_1^2 + M_2^2}, \qquad \cos\varepsilon = M_1/E, \qquad \sin\varepsilon = M_2/E \tag{2.6}$$

formulas (2.1) take the form

$$p_2^{(0)}(\tau) = E\cos(\omega\tau - \varepsilon), \qquad \gamma_2^{(0)}(\tau) = M_3\cos\tau \tag{2.7}$$

Using (2.7), and by (1.20), formulas

$$S_{11}^{(0)} = S_{11}(p_2^{(0)}, \dot p_2^{(0)}, \gamma_2^{(0)}, \dot\gamma_2^{(0)}), \qquad S_{21}^{(0)} = S_{21}(p_2^{(0)}, \dot p_2^{(0)}, \gamma_2^{(0)}, \dot\gamma_2^{(0)})$$

become

$$S_{11}^{(0)} = E^2 \{[a(\cos^2\varepsilon - 1/_2) + b\omega^2 A_1^{-2}(\sin^2\varepsilon - 1/_2) + \\ + 1/_2(b\omega^2 A_1^{-2} - a)\cos 2(\omega\tau - \varepsilon)\} - 2M_3[x_0'(1-\cos\tau) + y_0'\sin\tau] \quad (2.8)$$

$$S_{21}^{(0)} = M_3 E\{a\cos\varepsilon + 1/_2(b\omega A_1^{-1} - a)\cos[(\omega-1)\tau - \varepsilon] - \\ - 1/_2(b\omega A_1^{-1} + a)\cos[(\omega+1)\tau - \varepsilon]\}$$

Substituting (2.7) and (2.8) into formulas (1.22) and (1.23), respectively, we obtain for all the values of ω (with the exception of the case $\omega = 1/2$, which will be considered later) the expressions

$$F_2^{(0)} = M_1 L(\omega)\cos\omega\tau + M_2 L(\omega)\sin\omega\tau + \ldots, \qquad \Phi_2^{(0)} = M_3 N(\omega)\cos\tau + \ldots \quad (2.9)$$

where

$$L(\omega) = \omega^2[-(aM_1^2 + b\omega^2 A_1^{-2} M_2^2) + 1/_4(M_1^2 + M_2^2)(C_1 A_1^{-1} + \\ + 3a + b\omega^2 A_1^{-2})] + 2M_3\omega^2 x_0' - [z_0' a^{-1} + \varkappa_1\nu(1-\omega^2)]$$

$$N(\omega) = -(aM_1^2 + b\omega^2 A_1^{-2} M_2^2) - 1/_2(M_1^2 + M_2^2)[aB_1 + \\ + \omega^2 A_1^{-2}(1-b)] + 2M_3 x_0' - [z_0' b^{-1} - \varkappa_1\nu(1-\omega^2)] \quad (2.10)$$

From formulas (2.4) and (2.5) we obtain

$$g_k(T_0) = -\frac{1}{\omega}\int_0^{T_0} F_k^{(0)}(t_1)\sin\omega t_1\,dt_1, \qquad \dot{g}_k(T_0) = \int_0^{T_0} F_k^{(0)}(t_1)\cos\omega t_1\,dt_1 \quad (2.11)$$

$$h_k(T_0) = -\int_0^{T_0}\Phi_k^{(0)}(t_1)\sin t_1\,dt_1, \qquad \dot{h}_k(T_0) = \int_0^{T_0}\Phi_k^{(0)}(t_1)\cos t_1\,dt_1 \quad \begin{pmatrix} T_0 = 2\pi n \\ k = 2, 3\end{pmatrix}$$

Hence, by using (2.9) we have

$$g_2(T_0) = -\pi n\omega^{-1} M_2 L(\omega), \qquad \dot{g}_2(T_0) = \pi n M_1 L(\omega) \\ h_2(T_0) = 0, \qquad \dot{h}_2(T_0) = \pi n M_3 N(\omega) \quad (2.12)$$

The constants M_1, ωM_2 and M_3, which represent the initial conditions of the generating solution (2.1), the deviations $\beta_1(\mu)$, $\omega\beta_2(\mu)$ and $\beta_3(\mu)$ and the correction for the period α, must be found from the conditions for periodicity of the solutions $p_2(\tau, \mu)$, $\gamma_2(\tau, \mu)$ and their first derivatives; these conditions have the form

$$\Psi_1 = p_2(T_0 + \alpha, \mu) - p_2(0, \mu) = 0, \quad \Psi_2 = \dot{p}_2(T_0 + \alpha, \mu) - \dot{p}_2(0, \mu) = 0 \\ \Psi_3 = \gamma_2(T_0 + \alpha, \mu) - \gamma_2(0, \mu) = 0, \quad \Psi_4 = \dot{\gamma}_2(T_0 + \alpha, \mu) - \dot{\gamma}_2(0, \mu) = 0 \quad (2.13)$$

However, on the strength of the existence of the first integral (1.24) of system (1.21), the condition for periodicity $\Psi_3 = 0$ is not independent [6]. Indeed, writing the integral (1.24) in the form

$$\gamma_2{}^2(T_0 + \alpha, \mu) + \dot{\gamma}_2^2(T_0 + \alpha, \mu) + \mu(\ldots) = \gamma_2^2(0, \mu) + \dot{\gamma}_2^2\,(0, \mu) + \mu(\ldots)$$

and substituting formulas (2.13), we obtain, according to condition (2.2)

$$2\,(M_3 + \beta_3)\,\Psi_3 + \Psi_3^2 + \mu\varphi_1\,(\Psi_1, \Psi_2, \Psi_3, \Psi_4, \mu) = 0 \qquad (2.14)$$

in which φ_1 is an entire function of all its arguments, and, besides, $\varphi_1(0,\ 0,\ 0,\ \mu) = 0$. When $M_3 \neq 0$, which, as is shown later, is always the case, then by formula (2.14) $\Psi_3 = f(\Psi_1, \Psi_2, \Psi_4, \mu)$, where f is an entire function of all its arguments, and $f(0,\ 0,\ 0,\ \mu) = 0$. Consequentl we can see from (2.13) that condition $\Psi_3 = 0$ is automatically satisfied when the remaining conditions

$$\Psi_1 = \Psi_2 = \Psi_4 = 0 \qquad (2.15)$$

are satisfied.

Substituting the initial conditions (2.2) in the integral (1.24) when $\tau = 0$, we obtain the equations determining M_3 and β_3

$$M_3^2 + 2M_3\beta_3 + \beta_3^2 + 2\mu\nu M_3\,(M_1 + \beta_1) + \ldots = (\gamma_0'')^{-2} - 1$$

Assuming that γ_0'' is independent of μ, we obtain

$$M_3^2 = (\gamma_0'')^{-2} - 1, \qquad \beta_3^2 + 2M_3\beta_3 + 2\mu\nu M_3\,(M_1 + \beta_1) + \ldots = 0 \quad (2.16)$$

From equations (2.16) and from condition (1.3) it follows that (2.17)

$$0 < M_3 = (1 - \gamma_0''^2)^{1/2}\,(\gamma_0'')^{-1} < \infty, \qquad \beta_3 = -\,\mu\nu\,(M_1 + \beta_1) + \ldots$$

because γ_0'' is an arbitrary parameter, and M_3 is an arbitrary positive constant.

In this way, the periodic solution (2.3) depends on one arbitrary constant M_3 and on a function $\beta_3(\mu)$, vanishing when $\mu = 0$. This propert) does not depend on the form of α and occurs in all considered cases.

Expanding the right-hand sides of equations (2.13) in power series of α and retaining only the linear terms (neglecting even the terms $\mu^2\alpha$), we obtain the independent conditions of periodicity (2.15) in the form

$$\begin{gathered} p_2(T_0, \mu) - M_1 - \beta_1 + \alpha\omega(M_2 + \beta_2) = 0 \\ \dot{p}_2(T_0, \mu) - \omega(M_2 + \beta_2) - \alpha\omega^2(M_1 + \beta_1) = 0 \\ \dot{\gamma}_2(T_0, \mu) - \alpha(M_3 + \beta_3) = 0 \end{gathered} \qquad (2.18)$$

By the last equation, from (2.18) and by (2.17) and (2.3), we determine the function

$$\alpha = \mu^2 [\dot{H}_2(T_0) + \mu\dot{H}_3(T_0) + \ldots] / (M_3 + \beta_3) \qquad (T_0 = 2\pi n) \qquad (2.19)$$

We can conclude from it that by neglecting terms of the order α^2 and $\mu^2\alpha$ in system (2.18), we neglect also terms of the order μ^4.

Using (1.25) and (2.2) we shall now investigate those periodic solutions which arise when the fundamental amplitudes vanish, that is

$$M_1 = 0, \qquad M_2 = 0 \qquad (2.20)$$

Substituting relations (2.19), (2.20) and (2.3) in the first two equations (2.18) and cancelling μ^2, we obtain the system determining β_1 and β_2

$$G_2(T_0) + \mu G_3(T_0) + \omega\beta_2 [\dot{H}_2(T_0) + \mu\dot{H}_3(T_0) + \ldots] / (M_3 + \beta_3) + \\ + \mu^2(\ldots) = 0 \qquad (2.21)$$

$$\dot{G}_2(T_0) + \mu\dot{G}_3(T_0) - \omega^2\beta_1 [\dot{H}_2(T_0) + \mu\dot{H}_3(T_0) + \ldots] / (M_3 + \beta_3) + \\ + \mu^2(\ldots) = 0$$

Using (2.12) we can transform the above system into

$$\begin{gathered} -[L_1(\omega) - \omega^2 N_1(\omega)]\pi n\beta_2/\omega + \mu[G_3(T_0) + \ldots] = 0 \\ [L_1(\omega) - \omega^2 N_1(\omega)]\pi n\beta_1 + \mu[\dot{G}_3(T_0) + \ldots] = 0 \end{gathered} \qquad (2.22)$$

Here $L_1(\omega)$ and $N_1(\omega)$ are obtained from (2.10) by replacing M_1, M_2 and M_3 by β_1, β_2 and $M_3 + \beta_3$. By (1.13), (1.17) and (1.19), we obtain

$$L_1(\omega) - \omega^2 N_1(\omega) = W(\omega)(\beta_1^2 + \beta_2^2) - z_0' W_1(\omega) \qquad (2.23)$$

$$W(\omega) = (a-1)(a+b-2)/2b, \qquad W_1(\omega) = (3a + 3b - 4ab - 2)/ab \qquad (2.24)$$

From the condition that the z-axis has to be directed along the major or the minor axis of the ellipsoid of inertia of the body, it follows that for all the considered ω, $W(\omega) > 0$. We shall now show that for every value of $\omega \geqslant 1$ there exists only one pair of numbers a^* and b^* satisfying the condition $W_1(\omega) = 0$, and that for the values of $\omega < 1$, $W_1(\omega) \neq 0$.

Eliminating a^* from the system of two nonlinear equations (2.25)

$$3a^* + 3b^* - 4a^*b^* - 2 = 0, \qquad \omega^2 - (a^* - 1)(b^* - 1)/a^*b^* = 0$$

we find

$$\omega^2 = \frac{(1-b^*)^2}{3b^*(^2/_3 - b^*)}, \quad \text{or} \quad b_{1,2}^* = \frac{1+\omega^2 \pm \sqrt{\omega^2(\omega^2-1)}}{3\omega^2+1} \tag{2.26}$$

It is seen that the above formula is valid only when $\omega \geqslant 1$. Consequently, we have the following relation: (2.27)

$$0 < a^* \leqslant b^* < 1 \qquad \left(a^* = \frac{1+\omega^2-\sqrt{\omega^2(\omega^2-1)}}{3\omega^2+1}, \quad b^* = \frac{1+\omega^2+\sqrt{\omega^2(\omega^2-1)}}{3\omega^2+1}\right)$$

When $\omega = 1$, it follows from formulas (2.27) that $a^* = b^* = 1/2$, and for every value of $\omega > 1$ there exists a unique pair of distinct values of a^* and b^* satisfying (2.25). Besides, by (2.26)

$$^1/_2 < a^* < b^* < {^2/_3} \tag{2.28}$$

Assuming that

$$z_0' W_1(\omega) \neq 0 \tag{2.29}$$

we obtain from equations (2.22) the expansions of β_1 and β_2 in power series in μ.

To make an estimate of the order of the first terms of these expansions, we shall consider the quantities $C_3(T_0)$ and $\dot{C}_3(T_0)$ under conditions (2.20). Calculations show that $C_3(T_0) = \dot{C}_3(T_0) = 0$ when $\omega \neq 2$. This means [7] that the expansions of β_1 and β_2 in series of integral powers of μ begin from terms whose order is not higher than μ^2. Consequently, when ω is rational and does not equal 1, 2, 1/2, the first terms in the expansion of the periodic solution of the system (1.6) and the quantities α both under the conditions $z_0' \neq 0$, $a = a^*$, $b = b^*$, can be expressed in the following form:

$$p_1 = -\mu x_0'/bB_1 + \mu\varkappa_1 M_3 \cos\tau + \ldots, \quad q_1 = \mu y_0'/aA_1 + \mu\varkappa_2 A_1^{-1} M_3 \sin\tau + \ldots$$

$$r_1 = 1 - \mu^2 M_3 [x_0'(1-\cos\tau) + y_0' \sin\tau] + \ldots, \quad \gamma_1 = M_3\cos\tau + \ldots$$

$$\gamma_1' = -M_3 \sin\tau + \ldots \tag{2.30}$$

$$\gamma_1'' = 1 + \mu^2 \{[M_3(1-a)^{-1}x_0' + M_3^2 z_0'(A_1-1)(A_1+1)^{-1}] +$$
$$+ M_3(1-b)^{-1} y_0' \sin\tau - M_3(1-a)^{-1} x_0' \cos\tau -$$

$$- {}^1/_2\, M_3{}^2 z_0'(A_1 - 1)\,(A_1 + 1)^{-1} \cos 2\tau\} + \ldots$$

$$\alpha = 2\mu^2 \pi n\,(M_3 x_0' - z_0' + \ldots) \tag{2.31}$$

3. Let us consider periodic solutions of system (1.21) when $\omega = 2$. For this purpose we shall find $g_3(T_0)$ and $\dot{g}_3(T_0)$ under condition (2.20). Under the last conditions formula (1.20) will be rewritten as

$$S_{11}{}^{(0)} = -2M_3\,[x_0'\,(1 - \cos\tau) + y_0' \sin\tau], \qquad S_{12}{}^{(0)} = S_{21}{}^{(0)} = 0 \tag{3.1}$$

$$S_{22}{}^{(0)} = [a\varkappa M_3 + {}^1/_2\, M_3{}^2\,(a\varkappa_1 + bA_1{}^{-1}\varkappa_2)] + a^{-1}\, bA_1{}^{-1}\, y_0' M_3 \sin\tau -$$
$$- a\varkappa M_3 \cos\tau - {}^1/_2\, M_3{}^2\,(a\varkappa_1 + bA_1{}^{-1}\,\varkappa_2)\cos 2\tau$$

Substituting expression (3.1) into formulas (1.23) and (1.22), and retaining only terms with sin 2τ and cos 2τ, we obtain the relation

$$F_3{}^{(0)} = V_1 \cos 2\tau + V_2 \sin 2\tau + \ldots \tag{3.2}$$

Here

$$V_1 = -\frac{x_0' z_0' M_3{}^2}{6ab^2}(12b^2 - b - 1), \qquad V_2 = -\frac{y_0' z_0' M_3{}^2}{6a^2 b\,(1 - b)}(9ab^2 - 17ab +$$
$$+ 2b^2 + 4a - 3b + 1) \tag{3.3}$$

By (3.2) and (2.11) we obtain

$$g_3\,(2\pi) = -{}^1/_2\,\pi V_2, \qquad \dot{g}_3\,(2\pi) = \pi V_1 \qquad (T_0 = 2\pi) \tag{3.4}$$

Let us now consider formulas (3.3). Substituting in (3.3) $a = (1 - b)/(3b + 1)$ obtained from (1.13) when $\omega = 2$, we have

$$V_1 = -6M_3{}^2\, x_0'\, z_0'\, b^{-2}\,(1 - b)^{-1}\,(b - {}^1/_3)\,(b + {}^1/_4)\,(b + {}^1/_3) \tag{3.5}$$
$$V_2 = -{}^3/_2\, M_3{}^2\, y_0' z_0'\, b^{-1}\,(1 - b)^{-2}\,(b - {}^1/_3)\,(b + 5)\,(b + {}^1/_3)$$

From formula (3.5) and from (1.13) it follows that V_1 and V_2 vanish simultaneously if one of the following two conditions:

$$a = b = {}^1/_3, \qquad x_0' = y_0' = 0 \tag{3.6}$$

is satisfied (from (2.22) it follows that $z_0' \neq 0$).

This means that, if any one of the two conditions (3.6) is satisfied, then the case considered reduces to the previous one and the first terms of the expansions of the periodic solution of system (1.6) and the α's can be obtained from formulas (2.30) and (2.31).

If V_1 and V_2 do not vanish simultaneously, then by substituting formulas (3.4) into system (2.22) and under conditions $W_1(2) \neq 0$ and $\omega = 2 (n = 1)$, we obtain

$$\beta_1 = \mu V_1 \,/\, z_0'\, W_1\,(2) + \ldots, \beta_2 = \mu V_2 \,/\, z_0'\, W_1\,(2) + \ldots$$

Consequently, in formulas (2.30) only the expressions for p_1, q_1 and γ_1'' will change and become (3.7)

$$p_1 = -\mu\frac{x_0'}{bB_1} + \mu\varkappa_1 M_3\cos\tau + \mu\frac{V_1}{z_0'W_1(2)}\cos 2\tau + \mu\frac{V_2}{z_0'W_1(2)}\sin 2\tau + \ldots$$

$$q_1 = \mu\frac{y_0'}{aA_1} + \mu\frac{\varkappa_2 M_3}{A_1}\sin\tau + \mu\frac{2V_1}{A_1 z_0'W_1(2)}\sin 2\tau - \mu\frac{2V_2}{A_1 z_0'W_1(2)}\cos 2\tau + \ldots$$

$$\gamma_1'' = 1 + \mu^2 M_3(s_0 + s_1\sin\tau + s_2\cos\tau + s_3\cos 2\tau + s_4\cos 3\tau + s_5\sin 3\tau) + \ldots$$

where

$$s_0 = \frac{x_0'}{1-a} + \frac{1}{2}M_3 z_0'\frac{A_1-1}{A_1+1} + \frac{aV_1}{z_0'W_1(2)}, \qquad s_1 = \frac{y_0'}{1-b} + \frac{V_2}{2z_0'W_1(2)}\left(\frac{2b}{A_1} - a\right)$$

$$s_2 = -\frac{x_0'}{1-a} + \frac{V_1}{2z_0'W_1(2)}\left(\frac{2b}{A_1} - a\right), \qquad s_3 = -\frac{1}{2}M_3 z_0'\frac{A_1-1}{A_1+1}$$

$$s_4 = -\frac{V_1}{2z_0'W_1(2)}\left(\frac{2b}{A_1} + a\right), \qquad s_5 = -\frac{V_2}{2z_0'W_1(2)}\left(\frac{2b}{A_1} + a\right)$$

Under conditions $W_1(2) = 0$ ($a^* = 1/13(5 - \sqrt{12})$ and $b^* = 1/13(5 + \sqrt{12})$) system (2.22) will take the form

$$\beta_1(\beta_1^2 + \beta_2^2) = -\mu V_1^*/W^*(2) + \ldots, \quad \beta_2(\beta_1^2 + \beta_2^2) = -\mu V_2^*/W^*(2) + \ldots \qquad (3.8)$$

Hence

$$\beta_1 = \mu^{1/3}Y_1(\mu), \qquad Y_1(\mu) = -V_1^*[W^*(2)(V_1^{*2} + V_2^{*2})]^{-1/3} + \mu^{1/3}(\ldots)$$

$$\beta_2 = \mu^{1/3}Y_2(\mu), \qquad Y_2(\mu) = -V_2^*[W^*(2)(V_1^{*2} + V_2^{*2})]^{-1/3} + \mu^{1/3}(\ldots) \qquad (3.9)$$

In this case the first terms of the expansion of the periodic solution of system (1.6) and the quantity α can be expressed in the following form:

$$p_1 = \mu^{1/3}Y_1(\mu)\cos 2\tau + \mu^{1/3}Y_2(\mu)\sin 2\tau - \mu x_0'/b^*B_1^* + \mu\varkappa_1^* M_3\cos\tau + \ldots$$

$$q_1 = \mu^{1/3}2Y_1(\mu)(A_1^*)^{-1}\sin 2\tau - \mu^{1/3}2Y_2(\mu)(A_1^*)^{-1}\cos 2\tau + \mu y_0'/a^*A_1^* + \mu\varkappa_2^*(A_1^*)^{-1}\sin\tau + \ldots$$

$$r_1 = 1 - \mu^2 M_3[x_0'(1-\cos\tau) + y_0'\sin\tau] + \ldots$$

$$\gamma_1 = (M_3 + \beta_3)\cos\tau + \mu^{4/3}\nu^*Y_1(\mu)\cos 2\tau + \mu^{4/3}\nu^*Y_2(\mu)\sin 2\tau + \ldots \qquad (3.10)$$

$$\gamma_1' = -(M_3 + \beta_3)\sin\tau - \mu^{4/3}2\nu_2^*Y_1(\mu)\sin 2\tau + \mu^{4/3}2\nu_2^*Y_2(\mu)\cos 2\tau + \ldots$$

$$\gamma_1'' = 1 + \mu^{4/3}M_3[a^*Y_1(\mu) + {}^1/_2 Y_1(\mu)(2b^*/A_1^* - a^*)\cos\tau + {}^1/_2 Y_2(\mu)(2b^*/A_1^* - a^*)\sin\tau - {}^1/_2 Y_1(\mu)(2b^*/A_1^* + a^*)\cos 3\tau - {}^1/_2 Y_2(\mu)(2b^*/A_1^* + a^*)\sin 3\tau] + \ldots$$

$$\alpha = \mu^2\pi\{2M_3 x_0' - 2z_0' - \mu^{1/3}[Y_1^2(\mu)(a^* + {}^1/_2 a^*B_1^* + 2(A_1^*)^{-2} - 2b^*(A_1^*)^{-2}) + Y_2^2(\mu)(4b^*(A_1^*)^{-2} + {}^1/_2 a^*B_1^* + 2(A_1^*)^{-2} - 2b(A_1^*)^{-2})]\} + \ldots \qquad (3.11)$$

Stars indicate the substitution $a = a^*$, $b = b^*$.

4. In order to determine the periodic solutions of system (2.21) when

$\omega = 1/2$, we obtain from formulas (1.23) and (1.22)

$$F_2^{(0)} = [(L(^1/_2) + x_0' M_3 L_2) M_1 + y_0' M_3 L_3 M_2] \cos {^1/_2} \tau + \\ + [(L(^1/_2) - x_0' M_3 L_2) M_2 + y_0' M_3 L_3 M_1] \sin {^1/_2} \tau + \ldots \tag{4.1}$$

$$\Phi_2^{(0)} = M_3 N(^1/_2) \cos \tau + \ldots, \quad L_2 = -{^1/_2} A_1 (a/b - {^1/_2} A_1^{-1}), \quad L_3 = {^1/_2}(1 - {^1/_2} A_1^{-1}) a^{-1} \tag{4.2}$$

From formulas (2.11) and (4.1) we find the expressions

$$G_2(T_0) = -4\pi [(L_1(^1/_2) - x_0'(M_3 + \beta_3) L_2) \beta_2 + y_0'(M_3 + \beta_3) L_3 \beta_1]$$
$$\dot{G}_2(T_0) = 2\pi [(L_1(^1/_2) + x_0'(M_3 + \beta_3) L_2) \beta_1 + y_0'(M_3 + \beta_3) L_3 \beta_2] \tag{4.3}$$
$$H_2(T_0) = 0, \qquad \dot{H}_2(T_0) = 2\pi (M_3 + \beta_3) N_1(^1/_2) \qquad (T_0 = 4\pi)$$

which, when substituted into the system (2.21), give the equations for β_1 and β_2

$$[L_1(^1/_2) - {^1/_4} N_1(^1/_2) - x_0' M_3 L_2] \beta_2 + y_0' M_3 L_3 \beta_1 + \mu [-{^1/_4}\pi^{-1} g_3(T_0) + \\ + \rho_1(\beta_1, \beta_2)] + \mu^2(\ldots) = 0 \tag{4.4}$$

$$[\dot{L}_1(^1/_2) - {^1/_4} N_1(^1/_2) + x_0' M_3 L_2] \beta_1 + y_0' M_3 L_3 \beta_2 + \mu [{^1/_2}\pi^{-1} \dot{g}_3(T_0) + \rho_2(\beta_1, \beta_2)] + \\ + \mu^2(\ldots) = 0$$

By formula (2.23), and under conditions $g_3(4\pi) = \dot{g}_3(4\pi) = 0$ which are satisfied when $\omega = 1/2$, system (4.4) can be expressed in the form

$$y_0' M_3 L_3 \beta_1 - [x_0' M_3 L_2 + z_0' W_1(^1/_2)] \beta_2 + W(^1/_2) \beta_2 (\beta_1^2 + \beta_2^2) + \mu \rho_3(\beta_1, \beta_2) + \\ + \mu^2(\ldots) = 0 \tag{4.5}$$

$$[x_0' M_3 L_2 - z_0' W_1(^1/_2)] \beta_1 + y_0' M_3 L_3 \beta_2 + W(^1/_2) \beta_1 (\beta_1^2 + \beta_2^2) + \mu \rho_4(\beta_1, \beta_2) + \\ + \mu^2(\ldots) = 0$$

where $\rho_i(\beta_1, \beta_2)$ $(i = 1, 2, 3, 4)$ are entire functions of their arguments vanishing at $\beta_1 = \beta_2 = 0$. We shall assume that the following condition:

$$M_3^2 (x_0'^2 L_2^2 + y_0'^2 L_3^2) - z_0' W_1^2(^1/_2) \neq 0 \tag{4.6}$$

is satisfied.

By the inequality $a > b > 1$, which results from the condition $\omega = 1/2$, we obtain from formulas (4.2) that $L_2 > 0$, $L_3 > 0$; besides, $W_1(1/2) \neq 0$.

Consequently, for any selection of numbers x_0', y_0' and z_0' satisfying the condition

$$z_0'(x_0'^2 + y_0'^2) \neq 0 \tag{4.7}$$

and any selection of the quantities a and b satisfying the relation

$\omega = 1/2$, there exists only one value M_3

$$M_3^\circ = [z_0' W_1^2 (^1/_2) / (x_0'^2 L_2^2 + y_0'^2 L_3^2)]^{1/2} \tag{4.8}$$

or, by (2.17) and (1.3), one value γ_0''

$$\Gamma_0'' = (x_0'^2 L_2^2 + y_0'^2 L_3^2)^{1/2} [x_0'^2 L_2^2 + y_0'^2 L_3^2 + z_0'^2 W_1^2 (^1/_2)]^{-1/2} \tag{4.9}$$

at which condition (4.6) is not satisfied. If condition (4.7) is not satisfied, relation (4.6) is satisfied for any value of M_3. In this way condition (4.6) is equivalent to the condition

$$\gamma_0'' \neq \Gamma_0'' \quad (\text{ or } M_3 \neq M_3^\circ) \tag{4.10}$$

When condition (4.10) is satisfied, then from equations (4.5) it follows [7] that β_1 and β_2, expanded in power series in μ, begin from a term of order not higher than μ^2. In this case, when $\omega = 1/2$, the first terms of the power series expansion of μ of the periodic solution of system (1.6) and the α's can be found from formulas (2.30) and (2.31).

5. Let us consider the case $\omega = 1$, which corresponds to the disc (1.15). In this case we must set in formulas (1.18) to (1.23)

$$\varkappa_1 = \varkappa_2 = \nu = 0, \qquad \nu_2 = B_1 = -1, \qquad A_1 = 1 \tag{5.1}$$

Then, substituting in formulas (1.22) and (1.23) for $S_{ij}^{(0)}$ $(i, j = 1, 2)$

$$\begin{aligned}
S_{11}^{(0)} &= -2M_3 [x_0' (1 - \cos\tau) + y_0' \sin\tau] - E^2 (a-b) [^1/_2 \cos 2(\tau - \varepsilon) + {}^1/_2 - \cos^2\varepsilon] \\
S_{12}^{(0)} &= E \{y_0' (1 - b/a) [\sin(\tau - \varepsilon) + \sin\varepsilon] - ab^{-1} x_0' [\cos(\tau - \varepsilon) - \cos\varepsilon]\} \qquad (5.2) \\
S_{21}^{(0)} &= {}^1/_2 M_3 E [-\cos(2\tau - \varepsilon) + \cos\varepsilon] \\
S_{22}^{(0)} &= M_3 (-ab^{-1} x_0' \cos\tau + ba^{-1} y_0' \sin\tau + ab^{-1} x_0') + E^2 b [^1/_2 \cos 2(\tau - \varepsilon) - {}^1/_2 + \sin^2\varepsilon]
\end{aligned}$$

and also by (2.11) and (2.20) we obtain

$$\begin{aligned}
G_2(T_0) &= -\pi\beta_2 [2(M_3 + \beta_3) x_0' + {}^1/_2 (a-b)(\beta_2^2 - \beta_1^2)] \qquad (5.3) \\
\dot G_2(T_0) &= \pi\beta_1 [2(M_3 + \beta_3) x_0' + {}^1/_2 (a-b)(\beta_2^2 - \beta_1^2)] \\
G_3(T_3) &= -\pi \{-2\beta_2 [y_0' (1 - a^{-1} b) \beta_2 + ab^{-1} x_0' \beta_1] + 3a^{-1} b^{-1} x_0' y_0' (M_3 + \beta_3)\} \\
\dot G_3(T_0) &= \pi \{-2\beta_1 [y_0' (1 - ba^{-1}) \beta_2 + ab^{-1} x_0' \beta_1] + [a^{-2} (a+1) y_0'^2 - \\
&\qquad - b^{-2} (b+1) x_0'^2] (M_3 + \beta_3)\} \\
H_2(T_0) &= 0, \quad \dot H_2(T_0) = \pi (M_3 + \beta_3) [2(M_3 + \beta_3) x_0' - (a\beta_1^2 + b\beta_2^2)] \qquad (T_0 = 2\pi)
\end{aligned}$$

Substituting formulas (5.3) in system (2.21) and in (2.19) when $\omega = 1$, we obtain the equations determining β_1 and β_2 and the expression for α in the form

$$\beta_1 (\beta_1^2 + \beta_2^2) = -\mu V_3 + \ldots, \quad \beta_2 (\beta_1^2 + \beta_2^2) = -\mu V_4 + \ldots \tag{5.4}$$

$$V_3 = 2M_3 [a^{-2} (a+1) y_0'^2 - b^{-2} (b+1) x_0'^2], \qquad V_4 = 6M_3 a^{-1} b^{-1} x_0' y_0' \tag{5.5}$$

$$\alpha = \mu^2\pi\,(2M_3x_0' - a\beta_1^2 - b\beta_2^2 + \ldots) \tag{5.6}$$

Whence

$$\beta_1 = \mu^{1/3}Y_3(\mu), \qquad Y_3(\mu) = -V_3\,[V_3^2 + V_4^2]^{-1/3} + \mu^{1/3}(\ldots) \tag{5.7}$$
$$\beta_2 = \mu^{1/3}Y_4(\mu), \qquad Y_4(\mu) = -V_4\,[V_3^2 + V_4^2]^{-1/3} + \mu^{1/3}(\ldots)$$

From formulas (5.5) it follows that the quantities V_3 and V_4 do not vanish simultaneously; therefore the relations (5.7) are always true, with the exception of the case $x_0' = y_0' = z_0' = 0$, which we have not considered. Consequently, when $\omega = 1$, the first terms of the expansion of the periodic solution of system (1.6) and the α's can be expressed in the form

$$p_1 = \mu^{1/3}Y_3(\mu)\cos\tau + \mu^{1/3}Y_4(\mu)\sin\tau + \mu x_0'/b + \ldots$$
$$q_1 = \mu^{1/3}Y_3(\mu)\sin\tau - \mu^{1/3}Y_4(\mu)\cos\tau + \mu y_0'/a + \ldots \tag{5.8}$$
$$r_1 = 1 - \mu^2M_3\,[x_0'(1-\cos\tau) + y_0'\sin\tau] + \ldots, \qquad \gamma_1 = (M_3 + \beta_3)\cos\tau + \ldots$$
$$\gamma_1' = -(M_3 + \beta_3)\sin\tau + \mu^{4/3}Y_3(\mu)\sin\tau - \mu^{4/3}Y_4(\mu)\cos\tau + \ldots$$
$$\gamma_1'' = 1 - {}^1/_2\,\mu^{4/3}M_3\,[Y_3(\mu)\cos 2\tau + Y_4(\mu)\sin 2\tau - Y_3(\mu)] + \ldots$$
$$\alpha = \mu^2\pi\,[2M_3x_0' - \mu^{2/3}aY_3^2(\mu) - \mu^{2/3}bY_4^2(\mu) + \ldots] \tag{5.9}$$

6. In the case when ω is irrational, on the strength of conditions (2.17) and (2.20), we must [8] search for a periodic solution of system (1.21) whose period is $2\pi + \alpha$, in the form

$$p_2(\tau, \mu) = \chi_1(\beta_3, \mu)\cos\omega\tau + \chi_2(\beta_3, \mu)\sin\omega\tau + \sum_{k=2}^{\infty} G_k(\tau)\mu^k \tag{6.1}$$
$$\gamma_2(\tau, \mu) = (M_3 + \beta_3)\cos\tau + \sum_{k=2}^{\infty} H_k(\tau)\mu^k$$

where $\chi_1(\beta_3, \mu)$ and $\chi_2(\beta_3, \mu)$ are analytic functions to be determined; besides, $\chi_1(\beta_3, 0) = \chi_2(\beta_3, 0) = 0$. Writing these functions as the series

$$\chi_i(\beta_3, \mu) = \sum_{k=1}^{\infty}\left(Q_k^{(i)} + \frac{\partial Q_k^{(i)}}{\partial M_3}\beta_3 + {}^1/_2\frac{\partial^2 Q_k^{(i)}}{\partial M_3^2}\beta_3^2 + \ldots\right)\mu^k \qquad (i = 1, 2) \tag{6.2}$$

and replacing $M_1 + \beta_1$ and $M_2 + \beta_2$ by the above functions under the conditions of periodicity (2.18), we obtain by (6.1) an infinite system of equations determining the coefficients $Q_k^{(i)}$. Since the coefficients $Q_1^{(i)}$ are determined from the system of linear homogeneous algebraic equations

$$Q_1^{(1)}(1 - \cos 2\pi\omega) - Q_1^{(2)}\sin 2\pi\omega = 0, \qquad Q_1^{(1)}\sin 2\pi\omega + Q_1^{(2)}(1 - \cos 2\pi\omega) = 0$$

with non-zero determinant $\Delta = 2(1 - \cos 2\pi\omega)$, therefore they are trivial solutions of the above system, and series (6.2) begins from a term of order not lower than μ^2. It follows that the expansion of $p_2(\tau, \mu)$ in power series in μ also begins from a term of order not lower than μ^2; the quantities $H_2(T_0)$ and $\dot{H}_2(T_0)$ ($T_0 = 2\pi$) can be found from formula (2.12) by replacing $N(\omega)$ by $N_1(\omega)$ and M_3 by $M_3 + \beta_3$.

In this way the first terms of the expansion in power series of μ of the periodic solution of system (1.6) and the quantity α can be found from formulas (2.30) and (2.31).

7. We shall use now the Eulerian angles θ, ψ and φ when analyzing the obtained motion of a heavy solid about a fixed point

$$\cos\theta = \gamma'', \qquad \frac{d\psi}{dt} = \frac{p\gamma + q\gamma'}{1 - \gamma''^2}, \qquad \frac{d\varphi}{dt} = r - \frac{d\psi}{dt}\cos\theta, \qquad \tan\varphi_0 = \frac{\gamma_0}{\gamma_0'} \tag{7.1}$$

Let us mention in advance that, since the initial system (1.1) and system (1.21) are autonomous, the periodic solutions will remain periodic if t is replaced by $t + h$, where h is arbitrary. By formula (7.1), and by taking into account that the initial instant of time corresponds to the instant $t = h$, we can replace h by

$$\varphi_0 = {}^1/_2\,\pi + r_0 h + \ldots \tag{7.2}$$

introducing in this way an arbitrary initial angle of spin.

We have obtained previously four sets of formulas, (2.30), (3.7), (3.10) and (5.8), which exhaust all forms of different expansions of the periodic solutions with the prescribed approximations and restrictions. Substituting in (7.1) expansions (2.30), (3.7), (3.10) and (5.3), in which t has been replaced by $t + h$, and relations (1.5) and by using (2.17) in the form

$$M_3 = \tan\theta_0 \tag{7.3}$$

we obtain the following expressions for the angles θ and ψ. In the set (2.30)

$$\theta - \theta_0 = -\mu^2\,[\theta^{(1)}(t + h) - \theta^{(1)}(h)] + \ldots \tag{7.4}$$

$$r_0(\psi - \psi_0) = -\,\mathrm{Mg}C^{-1}z_0 t + \mu c\,\mathrm{cosec}\,\theta_0 \sqrt{\cos\theta_0}\,[\psi^{(1)}(t + h) - \psi^{(1)}(h)] + \ldots$$

$$\theta^{(1)}(t) = y_0' a^{-1} A_1^{-1}\sin r_0 t + x_0' b^{-1} B_1^{-1}\cos r_0 t - {}^1/_2\, z_0' \tan\theta_0\,(A_1 - 1)(A_1 + 1)^{-1}\cos 2r_0 t$$

$$\psi^{(1)}(t) = -\,x_0' b^{-1} B_1^{-1}\sin r_0 t + y_0' a^{-1} A_1^{-1}\cos r_0 t + {}^1/_4 \tan\theta_0\,(\varkappa_1 + \varkappa_2 / A_1)\sin 2r_0 t$$

In the set (3.7)

$$\theta - \theta_0 = -\mu^2\,[\theta^{(2)}(t + h) - \theta^{(2)}(h)] + \ldots \tag{7.5}$$

$$r_0(\psi - \psi_0) = -\,\mathrm{Mg}\,C^{-1} z_0 t + \mu c\,\mathrm{cosec}\,\theta_0 \sqrt{\cos\theta_0}\,[\psi^{(2)}(t + h) - \psi^{(2)}(h)] + \ldots$$

$$\theta^{(2)}(t) = s_1 \sin r_0 t + s_2 \cos r_0 t + s_3 \cos 2r_0 t + s_4 \cos 3r_0 t + s_5 \sin 3r_0 t$$

$$\psi^{(2)}(t) = -\left[\frac{x_0'}{bB_1} - \left(\frac{1}{2} - \frac{1}{A_1}\right)\frac{V_1}{z_0'W_1(2)}\right]\sin r_0 t + \left[\frac{y_0'}{aA_1} - \left(\frac{1}{2} - \frac{1}{A_1}\right)\frac{V_2}{z_0'W_1(2)}\right]\cos r_0 t +$$
$$+ \frac{\tan\theta_0}{4}\left(\varkappa_1 + \frac{\varkappa_2}{A_1}\right)\sin 2r_0 t + \frac{1}{3}\left(\frac{1}{2} + \frac{1}{A_1}\right)\frac{V \sin 3r_0 t}{z_0'W_1(2)} - \frac{1}{3}\left(\frac{1}{2} + \frac{1}{A_1}\right)\frac{V_2 \cos 3r_0 t}{z_0'W_1(2)}$$

In the set (3.10)

$$\theta - \theta_0 = -\mu^{4/3}[\theta^{(3)}(t+h) - \theta^{(3)}(h)] + \ldots \qquad (7.6)$$

$$r_0(\psi - \psi_0) = \{-Mgz_0C^{-1} + {}^1/_2\mu^{2/3}[Y_1^2(\mu) + Y_2^2(\mu)](\nu^* - 4\nu_2^*/A_1^*)\}t +$$
$$+ \mu^{1/3} c \operatorname{cosec}\theta_0 \sqrt{\cos\theta_0}\,[\psi^{(3)}(t+h) - \psi^{(3)}(h)] + \ldots$$

$$\theta^{(3)}(t) = {}^1/_2 Y_1(0)(2b^*/A_1^* - a^*)\cos r_0 t + {}^1/_2 Y_2(0)(2b^*/A_1^* - a^*)\sin r_0 t -$$
$$- {}^1/_2 Y_1(0)(2b^*/A_1^* + a^*)\cos 3r_0 t - {}^1/_2 Y_2(0)(2b^*/A_1^* + a^*)\sin 3r_0 t$$

$$\psi^{(3)}(t) = ({}^1/_2 - 1/A_1^*)Y_1(0)\sin r_0 t - ({}^1/_2 - 1/A_1^*)Y_2(0)\cos r_0 t +$$
$$+ {}^1/_3({}^1/_2 + 1/A_1^*)Y_1(0)\sin 3r_0 t - {}^1/_3({}^1/_2 + 1/A_1^*)Y_2(0)\cos 3r_0 t$$

In the set (5.8)

$$\theta - \theta_0 = -{}^1/_2\mu^{4/3}[\theta^{(4)}(t+h) - \theta^{(4)}(h)] + \ldots \qquad (7.7)$$

$$r_0(\psi - \psi_0) = {}^1/_2\mu^{1/3} c \operatorname{cosec}\theta_0 \sqrt{\cos\theta_0}\,[\psi^{(4)}(t+h) - \psi^{(4)}(h)] + \ldots$$

$$\theta^{(4)}(t) = -Y_3(0)\cos 2r_0 t - Y_4(0)\sin 2r_0 t - \quad \psi^{(4)}(t) = Y_3(0)\sin 2r_0 t - Y_4(0)\cos 2r_0 t$$

In all the cases φ is expressed by the formula

$$\varphi - \varphi_0 = [r_0 + MgC^{-1}r_0^{-1}(z_0\cos\theta_0 - x_0\sin\theta_0)]\,t + \ldots \qquad (7.8)$$

In all these formulas θ_0, as indicated by formula (7.3), is an arbitrary initial (at $t = 0$) angle of nutation; ψ_0 is an arbitrary initial angle of precession. Replacing h by φ_0 we can see from (7.2) that the expressions for the Eulerian angles θ, ψ and φ depend on four arbitrary constants θ_0, ψ_0, φ_0 and r_0 (r_0 is large).

From formulas (7.4) to (7.6) it follows that for a heavy solid, when $z_0 \neq 0$ (with the exception of the special cases discussed), with one point fixed, which spins rapidly about the major or the minor axis of the ellipsoid of inertia, it is possible to show initial conditions at which the solid will perform, in the first approximation, a pseudo-regular precession about the vertical axis. Four out of these six initia conditions, θ_0, ψ_0, φ_0 and r_0 (r_0 is large), can be arbitrary.

As an example, we shall consider the case of a regular precession of a rapidly spinning Lagrange gyroscope ($A = B$, $x_0 = y_0 = 0$). System (1.1) rewritten for this case, will have four particular solutions

$$\gamma'' = \gamma_0'', \qquad p = \lambda\gamma, \qquad q = \lambda\gamma', \qquad r = r_0 \tag{7.9}$$

$$\lambda = \frac{r_0(m_1 - 1)}{2\gamma_0''}\left[-1 + \left(1 + \frac{4\gamma_0'' c_1^2}{r_0^2 (m_1 - 1)^2}\right)^{1/2}\right], \qquad m_1 = \frac{A - C}{A}, \qquad c_1^2 = \frac{Mgz_0}{A}$$

Substituting solutions (7.9) into (7.1), we obtain the relations

$$\cos\theta = \gamma_0'', \qquad d\psi / dt = \lambda, \qquad d\varphi / dt = r_0 - \lambda\gamma''_0$$

for which, taking into account that r_0 is large, we obtain

$$\theta = \theta_0, \qquad \psi = \psi_0 - \mathrm{Mg}\, C^{-1} r_0^{-1} z_0 t + \ldots, \varphi = \varphi_0 + (r_0 + \mathrm{Mg}C^{-1} r_0^{-1} z_0 \cos\theta_0)t + \ldots \tag{7.10}$$

Expressions (7.10) coincide with formulas (7.4), (7.5), (7.6) and (7.8) when they are rewritten for the considered case.

Formulas (7.7) are for the case $\omega = 1$, under the condition that $z_0 = 0$. Under this last condition the restrictions caused by retaining only the first terms in the expansion of the required periodic solution permit also an investigation of the cases of irrational ω and of $\omega = 1/2$. Formula (7.4) takes care of these two cases. Formulas (7.4) and (7.7) will be rewritten now in the following form:

$$\theta - \theta_j = R_j \sin\theta_0 \cos(jr_0 t - \varepsilon_j) + \ldots \quad (j = 1, 2), \quad R_1 = \mu^2 E_1 \operatorname{cosec}\theta_0$$

$$\psi - \psi_j = R_j \sin(jr_0 t - \varepsilon_j) + \ldots, \qquad R_2 = {}^1\!/_2 \mu^{4/3} E_2 \operatorname{cosec}\theta_0 \tag{7.11}$$

$$\varphi - \varphi_0 = (r_0 - \mathrm{Mg}C^{-1} r_0^{-1} x_0 \sin\theta_0)\, t + \ldots$$

$$E_1 = \left[\left(\frac{x_0'}{a - 1}\right)^2 + \left(\frac{y_0'}{b - 1}\right)^2\right]^{1/2}, \qquad E_2 = [Y_3^2(0) + Y_4^2(0)]^{1/2}, \qquad \varepsilon_j = \varepsilon_j^\circ - j(\varphi_0 - {}^1\!/_2\pi)$$

$$\cos\varepsilon_1^\circ = \frac{x_0'}{(1 - a) E_1}, \quad \sin\varepsilon_1^\circ = \frac{y_0'}{(b - 1) E_1}, \quad \cos\varepsilon_2^\circ = \frac{Y_3(0)}{E_2} \quad \sin\varepsilon_2^\circ = \frac{Y_4(0)}{E_2}$$

$$\theta_j = \theta_0 - R_j \sin\theta_0 \cos\varepsilon_j, \qquad \psi_j = \psi_0 + R_j \sin\varepsilon_j$$

where $j = 2$ is for the case of a disc.

Let us consider a spherical rectangle bounded by parallels distant from the middle parallel θ_j by the angle $\pm R_j \sin\theta_0$, and by meridians distant from the middle meridian ψ_j by the angle $\pm R_j$. Then the trajectory of the z-axis of our solid will be the ellipse

$$\frac{(\theta - \theta_j)^2}{R_j^2 \sin^2\theta_0} + \frac{(\psi - \psi_j)^2}{R_j^2} = 1 \qquad (j = 1, 2) \tag{7.12}$$

tangent to the sides of this spherical rectangle at their midpoints.

The z-axis of the body tracing this ellipse performs, in the first approximation, a periodic motion whose period is $T = 2\pi / jr_0$ and at the instants of time

$$t_{n_1} = (\pi n_1 + \varepsilon_j) / j r_0 \qquad (n_1 = 0, \pm 1, \pm 2, \ldots)$$

it will pass through the points of intersection of the middle parallel and the bounding meridians, and at the instants

$$t_{n_2} = [{}^1/_2 (2n_2 + 1) \pi + \varepsilon_j] / j r_0 \qquad (n_2 = 0, \pm 1, \pm 2, \ldots)$$

it will pass through the points of intersection of the middle meridian with the bounding parallels. Formula (7.11) shows that the spin differs very little from the uniform rotation with large angular velocity r_0.

Thus, for a heavy solid with one point fixed, with $z_0 = 0$, with the described restrictions on the moments of inertia, which spins rapidly about the major or the minor axis of the ellipsoid of inertia, we can show initial conditions at which the body, in the first approximation, will perform the motion which we have just found. In this case, as in the case when $z_0 \neq 0$, out of the six initial conditions, at least four, θ_0, ψ_0, φ_0 and r_0 (r_0 is large) can be arbitrary.

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